Problem Set 5 – Relativity (`GOI36A`)

20/10/2020

1 Hands-on with Christoffel Symbols and the Riemann Tensor

1.) Consider a 3d Euclidean spacetime with coordinates (r, θ, ϕ), for which the metric is the usual metric in spherical coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \ . \tag{1.1}$$

First, without doing any calculations, try to identify what the non-trivial, independent Christoffel symbols are. By non-trivial, we mean that they are not obviously zero. And by independent, we mean that all other non-trivial symbols can be obtained from this set by symmetry. You should find that there are six such symbols.

Then, compute them explicitly.

2.) Next, again without doing any calculations, identify what the non-trivial, independent components of the Riemann tensor are. (You'll really have to make sure you understand the symmetries of the Riemann tensor to do this.) You should find that there are six independent components, from which we can obtain all other non-zero components via symmetries.

Then, compute them explicitly.

3.) If you have done all of this correctly, you'll notice something striking about the Riemann tensor for this spacetime. Physically, explain what's going on.

4.) Now, consider a 4d Minkowski spacetime, for which the metric in spherical coordinates (t, r, θ, ϕ) takes the form

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \ . \tag{1.2}$$

Compute the Christoffel symbols and Riemann tensor in this coordinate system. (Hint: don't work harder than you have to)

2 Curvature on the Two-Sphere

5.) Again, consider the 3d Euclidean spacetime in spherical coordinates, as presented in the first problem. Consider the surface defined by $r = L$, for some constant L. Write down the induced metric on this surface, i.e. the two-sphere S^2 with radius L.

6.) Compute the Christoffel symbols, the Riemann tensor, the Ricci tensor, and the Ricci scalar on this S^2. What is similar and what is different about these calculations when you compare them to the calculations you did in problems 1 and 2?

7.) Write down the geodesic equations explicitly on this S^2. Use these equations to argue that all geodesics on S^2 are (sections of) *great circles*, as depicted in figure 1.

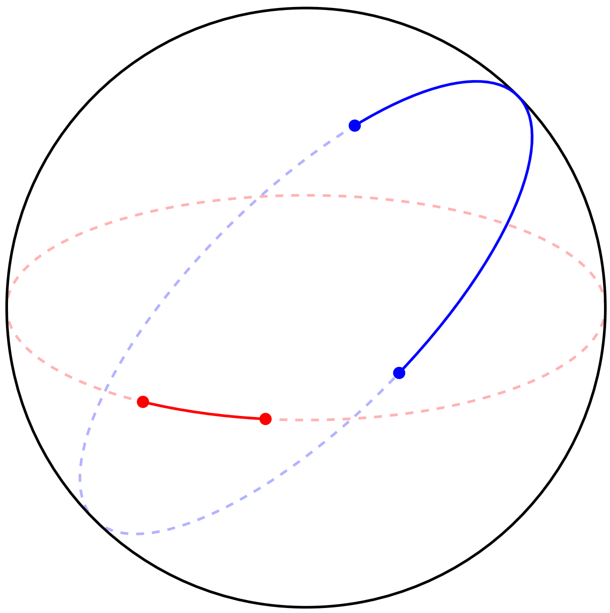

Figure 1. Sections (depicted by solid lines) of two different great circles (depicted by dashed lines). Great circles are defined as the intersection of the sphere and a plane going through the center of the sphere. The result is a closed circular loop that goes through antipodal points on the sphere.

Hint #1: try setting ϕ to a constant, or setting $\theta = \pi/2$, and then solving the geodesic equations. Try to understand the resulting geodesics physically, and see if you can generalize them.

Hint #2: is there anything special about the north and south pole on the two-sphere?

3 Practice Makes Perfect

8.) Consider a 2d spacetime for which the metric is

$$ds^2 = -f(r)dt^2 + h(r)dr^2 \ , \tag{3.1}$$

for some functions $f(r)$ and $h(r)$ Compute the corresponding Christoffel symbols, the Riemann tensor, the Ricci tensor, and the Ricci scalar. Then, write down the geodesic equation explicitly.

Note: the answer you're going to get isn't necessarily the prettiest thing in the world. That's okay, though. This problem is to get you experience with computing the Riemann tensor for a metric that is not completely specified. As we'll see later, this skill is useful when you're trying to solve the Einstein equation.

9.) Consider a flat, 1d spacetime with metric $ds^2 = dx^2$. Then, make a coordinate transformation to a new coordinate x', defined by

$$x = f(x') \ , \tag{3.2}$$

for some arbitrary function $f(x')$. Compute the metric and inverse metric in this new coordinate frame. Then, compute the Christoffel symbols and show that the geodesic equation is

$$\frac{d^2x'}{d\lambda^2} + \frac{f''(x')}{f'(x')}\left(\frac{dx'}{d\lambda}\right)^2 = 0 \ . \tag{3.3}$$

Lastly, try to think about the Riemann tensor. Can you compute it for this 1d spacetime?